Logarithms

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1. Introduction

In the early 1600s, a Scottish necromancer took two Greek words, smashed them together, and published in Latin. Ok, maybe he didn't *really* raise the dead. But I'd imagine that whatever pain would be involved in doing so pales in comparison to the horrors faced by math students when introduced to this Scotsman's word: **logarithm**.

Historically, logarithms were a revolutionary tool for making computations easier. Yet in my teaching experience, I've found many a student grow frustrated when needing to understand them. And I don't think I'll get much push-back for suggesting that logarithms are one of the more difficult topics for precalculus mathematics courses. My YouTube audience would seem to agree.

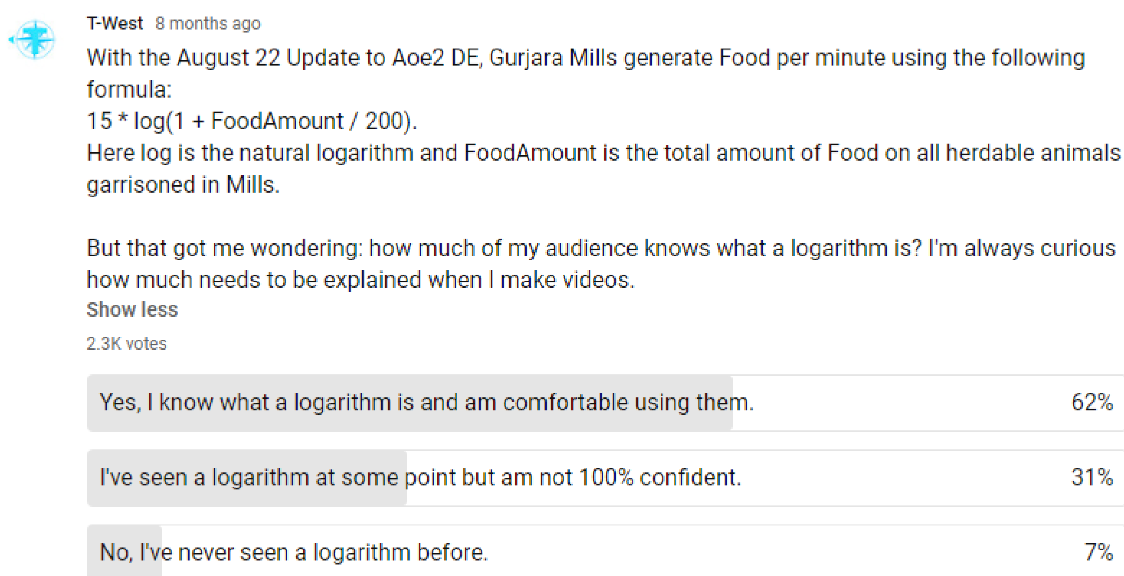

Figure 1: Looks like it's time for a lesson on logarithms!

But what exactly makes learning about them so difficult? Maybe it's the need to move beyond the basic operations of addition, subtraction, multiplication, and division. Possibly it's the usage of functions and other mathematical abstractions. Or perhaps it's the intimidating nature of the word itself.

Rather than discuss a topic with which a large percentage of viewers wouldn't be comfortable, I figured this would be a perfect opportunity to put together a math lesson—so welcome to "Topics That T-West Thinks Everyone Should Know About Logarithms!"

We'll cover the following:

- Where the word *logarithm* comes from and what it means.
- The relationship between logarithms and exponents.

- How logarithms turn difficult multiplications into easier additions.
- The intuition driving various logarithm formulas and properties.
- What is "natural" about the natural logarithm and the number e.
- And why logarithms result in "fast" algorithms and "slow" growth.

Oh, and we'll discuss some Scottish necromancy along the way.

1.1 About These Notes

These notes accompany a YouTube video about logarithms (the link will be posted when the video is uploaded). Throughout the text, segments of the video corresponding to the relevant topics are embedded. The notes themselves are availble online at twestaoe.net/math/logarithms. They're also available to download as a pdf (although the pdf is missing the interactive features available on the website).

In high school mathematics I'd imagine most students get by without ever reading the textbook. I know I mostly didn't. And to be fair, most of those books probably aren't great reads anyway. But as we advance in our mathematical careers, reading the book becomes a crucial part of the subject. And it can be difficult to learn how to read a math textbook—especially for an algebra or precalculus level where most students don't have any prior experience in doing do.

My hope is that by including videos along with these notes, they become easier to read. And thereby help not just to teach about logarithms, but also to serve as an introduction to reading mathematics. And since a large part of such reading involves solving problems, I've included a variety of exercises throughout the text. They should provide some fun and interesting puzzles to work through.

2. What Is a Logarithm?

Our journey begins in the late 1500s and early 1600s. Mathematicians of the era faced a key challenge, one with which I'm sure all of us are familiar: multiplication is harder than addition. What do I mean by "harder?" Well, consider adding two multiple-digit numbers: 243 and 142. The standard algorithm is to line these numbers up and add their pairs of digits individually from right to left.

$$\begin{array}{r} 243 \\ +\ 142 \\ \hline \end{array} \qquad \begin{array}{r} 243 \\ +\ 142 \\ \hline 5 \end{array} \qquad \begin{array}{r} 243 \\ +\ 142 \\ \hline 85 \end{array} \qquad \begin{array}{r} 243 \\ +\ 142 \\ \hline 385 \end{array}$$

Additions can get a bit more difficult if we need to "carry a 1," but overall this process is pretty straightforward.

Multiplication, on the other hand, involves more steps. Let's take 243 and 142 once again and multiply them.

$$\begin{array}{r} 243 \\ \cdot\ 142 \\ \hline \end{array} \qquad \begin{array}{r} 243 \\ \cdot\ 142 \\ \hline 486 \end{array} \qquad \begin{array}{r} 243 \\ \cdot\ 142 \\ \hline 486 \\ 9720 \end{array} \qquad \begin{array}{r} 243 \\ \cdot\ 142 \\ \hline 486 \\ 9720 \\ 24300 \end{array} \qquad \begin{array}{r} 243 \\ \cdot\ 142 \\ \hline 486 \\ 9720 \\ +\ 24300 \\ \hline 34506 \end{array}$$

Again we start by lining up the numbers. But now we need to multiply the top number by every digit in the bottom number. And we have to shift the results as we go along, then add everything together at the end.

Phew! That was a lot of work. It's nothing new or difficult, just rather tedious. But more than that, we have lots of opportunities for making mistakes. Keeping track of all of the digits and carries and shifts by hand requires a good deal of discipline and attention to detail. Wouldn't it be great if we didn't have to do all of this? Well, that's where logarithms are going to help us!

Logarithms first were created as a tool to avoid multiplications. As the centuries have gone on we've learned that logarithms have a myriad of other mathematical properties. But this is where we'll start: simple addition and multiplication. To begin, we'll reexamine some of the arithmetic we learned in elementary school—doing so will help us motivate the definition of a logarithm.

2.1 It All Starts With Counting

Let's think back to our early educations. First we learned about counting, then about addition and subtraction, and then about multiplication, where we were introducted to evaluating expressions such as $3 \cdot 5$. Now if your first instinct is to yell out "15," then congratulations—you memorized your times tables when you were a child! If you're anything like me, then you had to learn the products of single-digit number pairs.

$\cdot$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	2	6	9	12	15	18	21	24	27
4	0	2	8	12	16	20	24	28	32	36
5	0	2	10	15	20	25	30	25	40	45
6	0	2	12	18	24	30	36	42	48	54
7	0	2	14	21	28	35	42	49	56	63
8	0	2	16	24	32	40	48	56	64	72
9	0	2	18	27	36	45	54	63	72	89

But what if we didn't have this table? How would we know what to do when faced with a multiplication problem? Well, we probalby learned that multiplication is a "repeated addition." If we're given $3 \cdot 5$, we can write

$$3 \cdot 5 = \underbrace{3 + 3 + 3 + 3 + 3}_{5\text{ times}} = 15.$$

We add 3 to itself 5 times, and our answer is the total sum of 15. The multiplication consists of three components: a number, a count, and a total. To be a bit more general, we can write

$$\text{number} \cdot \text{count} = \underbrace{\text{number} + \text{number} + \cdots + \text{number}}_{\text{count times}} = \text{total}.$$

The count tells us how many times to add the number to itself in order to get the total.

As we take math classes, we're asked to solve "multiplication problems." These problems present us with two of the three components and endeavor us with discerning the third. In elementary school, we're given the number and the count, and we have to find the total. A question might be written as follows, where we have to fill in the box:

$$6 \cdot 4 = \boxed{}.$$

Our answer is the product 24:

$$6 \cdot 4 = \boxed{24}.$$

Later on in our education, we're asked another type of question. Instead of the number and the count, we're given the number and the total. We have to find the count and write it in the box:

$$6 \cdot \boxed{} = 24,$$

$$6 \cdot \boxed{4} = 24.$$

Or, instead of a box, the question may use a variable:

$$6n = 24.$$

Here we're asked to "solve for n." Which is just answering the question, "how many times do we add 6 to get 24?" Of course we could figure this out by "dividing both sides by 6:"

$$\begin{aligned} 6n &= 24, \\ \frac{\cancel{6}n}{\cancel{6}} &= \frac{24}{6}, \\ n &= 4. \end{aligned}$$

But let's think about how to solve this type of problem if we didn't know an algorithm for division. We could figure out the answer just by counting! We add 6 to itself until we get to 24, then count the number of 6's we need:

$$\begin{aligned} &6 = 6, \\ &6 + 6 = 12, \\ &6 + 6 + 6 = 18, \\ &\underbrace{6 + 6 + 6 + 6}_{4 \text{ times}} = 24. \end{aligned}$$

Our answer is $n = 4$.

The algorithms we learn for long division and for finding and cancelling out common factors are definitely useful. But it's important to remember that they're tools we use for finding the count. And if we wanted to, we could just do the counting ourselves.

2.2 Repeated Multiplication

As we advance further in school, following up after repeated additions, we have repeated multiplications. And we represent these repetitions using exponents. For example,

$$3^5 = \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{5 \text{ times}} = 243.$$

Now we probably don't have 3^5 memorized like we do our multiplication tables. But we can just multiply by 3 repeatedly to get the answer. We have the same three components as before: a number, a count, and a total. In general we can write

$$\text{number}^{\text{count}} = \underbrace{\text{number} * \text{number} * \cdots * \text{number}}_{\text{count times}} = \text{total}.$$

Here we use a $*$ instead of a $\cdot$ to denote multiplication, to avoid confusion with the center dots. In this context we call the number the *base*, the count the *exponent*, and the total the *product*:

$$\text{base}^{\text{exponent}} = \text{product}.$$

When you see the word *exponent*, think about the word *expose*. The term refers to the written appearance of the number as being exposed up and to the right, rather than to any particular mathematical property.

Just like with multiplication, "math problems" involving exponentiation give use two pieces of information and task us with finding the third. We start, once again, by being given a number and a count and being asked to find the total:

$$6^3 = \boxed{}.$$

And again, we can multiply 6 by itself 3 times to obtain the answer:

$$6^3 = \underbrace{6 \cdot 6 \cdot 6}_{3 \text{ times}} = \boxed{243}.$$

Now comes the more interesting question. Suppose, as we did previous with "multiplication problems," that we're given a number and a total. How do we "solve for the count?" Here we don't have the division algorithm to aide us like we did with our earlier analogous situation. But we still can resort to counting. Suppose we want to solve

$$2^n = 32.$$

We can do so by multiplying 2 by itself until we reach 32 while counting the number of multiplications:

$$\begin{aligned}
&2 = 2,\\
&2 \cdot 2 = 4,\\
&2 \cdot 2 \cdot 2 = 8,\\
&2 \cdot 2 \cdot 2 \cdot 2 = 16,\\
&\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{5 \text{ times}} = 32.
\end{aligned}$$

Thus our answer is $n = 5$. And just like we use the name *division* refer to the process of finding the count when working with repeated addition, we need a name for the concept of finding the count when working with repeated multiplication. And for that, it's time to introduce *logarithms*.

2.3 Defintion of a Logarithm

Let's dive right in and state the definition.

Definition 1 (Logarithm)**.** Let x and b be positive real numbers with b not equal to 1. If n is a number such that b multiplied by itself n times equals x—that is, such that $b^n = x$—we write

$$\log_b x = n.$$

This notation is read "the base b logarithm of x equals n," or more simply as "log base b of x is n." Here b is called the *base* of the logarithm, and x is called the *argument* of the logarithm.

We'll discuss more advanced properties and techniques for computing logarithms later. But for now let's just concentrate on what this definition is telling us.

When we have an exponential equation such as

$$2^5 = 32,$$

we have another notation for writing it:

$$\log_2 32 = 5.$$

This equation is read, "the base 2 logarithm of 32 equals 5." It tells us that if we multiply 2 by itself 5 times, we get 32.

Now for some logarithm examples. If we're given an equation and asked to find what the exponent is, we can do what we did previous: multiply the base by itself until we reach the product, then count the number of times we multiplied.

1. $\log_2 64$.

   To calculate this logarithm, write out products of 2 until we reach 64. Hence $\log_2 64 = 6$.

2.4 Some Caveats About the Definition

In our definition of $\log_b x$, we impose restrictions on the numbers b and x. We require that both of these are positive, and further that b is not equal to 1. What do these conditions mean, and why are they important?

We define the base b logarithm of x only when the the following conditions hold:

For example, what happens if b or x is zero? Well, $\log_2 0$ is not defined, since there is no power of 2 such that $2^n = 0$. Multiplying 2 by itself just gives bigger and bigger positive numbers, we can't multiply by 2 and get zero. Similarly, since $0 \cdot 0 = 0$, there is no n such that $0^n = 2$. Multiplying repeatedly by zero keeps yielding zero. Hence $\log_0 2$ is not defined either.

You'll find that we don't gain any utility from attempting to include other cases in our definition of logarithms, anyway. In a more advanced course we could study how complex numbers relate to logarithms. But for now, and for general-purpose usage of logarithms with real numbers, these restrictions suffice.

2.5 Exercises

3. The Inverse of Exponentiation

3.1 What Is an "Inverse?"

3.2 How Logarithms and Exponentiation "Undo" Each Other

3.3 But Isn't Taking Roots an "Inverse" of Exponentiation Too?

3.4 Exercises